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LETTER TO THE EDITOR

Hidden symmetry in a conservative equation for nonlinear growth

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Abstract. Growth equations for nonlinear deposition are considered from the perspective of fluid dynamics and the theory of turbulence. A nonlinear equation which conserves mass is found, contrary to current belief (Hwa T and Kardar M 1992 *Phys. Rev.* A **45** 7002), to possess shift symmetry.

In recent years it has become clear that a diverse range of complicated physical phenomena, such as interface growth or the dynamics of polymers in random media or the behaviour of flame fronts, can be usefully described by quite simple stochastic partial differential equations (PDES) [1, 2]. The processes by which such equations are obtained owe much to the subject of macroscopic fluid dynamics, while their subsequent analysis has to some extent been influenced by the theory of fluid turbulence [3–8]. In this letter we show that lessons learned from turbulence theory can also illuminate the formulation and choice of the governing PDES. In particular, a rigorous distinction between mean and fluctuating variables, and careful choice of appropriate frames of reference, can reveal hidden symmetries in conservative growth equations.

In order to concentrate on a specific physical problem, we shall choose the topic of nonlinear deposition. In particular, we consider a granular material being poured into a bin. As the material in the bin builds up, one is interested in the fluctuations in the height $H(\boldsymbol{x}, t)$ of the free surface, as measured from the base of the bin. The specification of the problem may be completed by representing the pouring process in terms of a source $S(\boldsymbol{x}, t)$, such that

$$S(\boldsymbol{x}, t) = \overline{S}(\boldsymbol{x}, t) + \eta(\boldsymbol{x}, t) \tag{1}$$

where $\overline{S}$ in the mean value of S and η is the fluctuation about the mean. It follows, of course, that $\langle \eta(\boldsymbol{x}, t)\rangle = 0$, where $\langle \ldots \rangle$ denotes the operation of taking means. It is usual to take η to have a Gaussian distribution, but this will not concern us here.

It follows from (1) that the instantaneous surface height may also be written as the sum of a mean and a fluctuation, thus

$$H(\boldsymbol{x}, t) = \overline{H}(\boldsymbol{x}, t) + h(\boldsymbol{x}, t) \tag{2}$$

where $\overline{H}$ is the mean, h is the fluctuation and again $\langle h\rangle = 0$. We should note that the restriction of the deposition process to a bin and $\overline{S}(\boldsymbol{x}, t) = \overline{S}(t)$ allows us to assume that the mean height does not depend on the spatial coordinates and hence that

$$\nabla \overline{H} = 0 \tag{3}$$

or $\overline{H}(\boldsymbol{x}, t) \equiv \overline{H}(t)$ only.

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On the basis of a 'blob' model, which ascribed some liquid properties to the motion of the powder, Edwards and Wilkinson [3] obtained linear Langevin equations for the Fourier modes of the surface growth. In real space, and in our present notation, these lead to the form

$$\frac{\partial H}{\partial t} = \nu \nabla^2 H + S(\boldsymbol{x}, t)$$

or, in the frame of reference of the mean interface,

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(\boldsymbol{x}, t) \tag{4}$$

where ν is a real constant and represents the effects of surface tension. We shall refer to this as the EW equation. These authors also discussed the possibility of introducing a nonlinear term of form $(\nabla h)^2$, but it was left to Kardar, Parisi and Zhang (KPZ) [4] to formally introduce a physical basis for such a term.

KPZ argued that surface growth would take place locally normal to the interface. When this effect is resolved in the vertical direction, it leads to a nonlinear contribution to δh which can be added on to the EW equation for time interval δt. The resulting equation is usually written as

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\boldsymbol{x}, t). \tag{5}$$

We note (a) that λ is another real constant; and (b) that this equation is described as being 'in the co-moving frame' [4].

Equation (5) is the famous KPZ equation and it has had an enormous influence on this whole subject. However, as is well known, it does not conserve mass. Hence its use may not be appropriate in applications where conservation of mass is believed to be a requirement. Indeed, as a general proposition, one might say that the more appropriately a discrete system can be modelled as being in the hydrodynamic limit, then arguably the more necessary it is that it should satisfy *all* the equations of hydrodynamics.

This aspect has been discussed by Hwa and Kardar [9], in the context of avalanches in sandpiles. They note that in hydrodynamics, conservation of mass takes the form of the continuity equation, viz.

$$\frac{\partial h}{\partial t} + \nabla \cdot \boldsymbol{j} = \eta \tag{6}$$

where $\boldsymbol{j}$ is the current or flux of h. A conservative alternative to the KPZ equation can be obtained by writing the current as

$$\boldsymbol{j} = -\nu \nabla h - \lambda h \nabla h \tag{7}$$

and substituting into equation (6), with the result

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda \nabla \cdot (h \nabla h) + \eta(\boldsymbol{x}, t). \tag{8}$$

This equation appears as equation (34) in the paper by Hwa and Kardar and it is worth quoting what they say about it, thus:

> 'It looks somewhat like the equation that describes the evolution of growing interfaces [i.e. the KPZ equation], but is in fact quite different because it does not have the symmetry $h \rightarrow h$ + constant.'

Perhaps for this reason, it seems that equation (8) has largely been ignored as a starting point for studies of nonlinear growth. However, it is our opinion that the model behind equation (8) does have shift symmetry and that it is merely a matter of distinguishing more clearly between mean and fluctuating variables in order to reveal this fact. To do this, we follow the standard procedure in turbulence (originally due to Reynolds in the late 1800s!). That is, we begin with the equation of motion for the instantaneous variable H and then make the decompositions (1) and (2). The method will be illustrated first by application to the KPZ equation, which we now write in terms of the instantaneous height. Clearly it must take the form

$$\frac{\partial H}{\partial t} = \nu \nabla^2 H + \frac{\lambda}{2}(\nabla H)^2 + S(\boldsymbol{x}, t). \tag{9}$$

Now we substitute (1) and (2) in equation (9), average all terms, and, recalling equation (3), obtain

$$\frac{\partial \overline{H}}{\partial t} = \frac{\lambda}{2}\overline{(\nabla h)^2} + \overline{S} \tag{10}$$

for the mean interface height. Also, subtracting (10) from (9), yields

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}\left[(\nabla h)^2 - \overline{(\nabla h)^2}\right] + \eta \tag{11}$$

for the fluctuating height $h(\boldsymbol{x}, t)$. Equation (10) shows the so-called excess velocity [1], where even when $\overline{S} = 0$ there is a transient mean velocity until the surface is flat. Equation (11) is the correct form of the KPZ equation in a frame moving with the mean interface. Note that both sides of equation (11) vanish when averaged. Normally in the literature equation (5) represents both the mean and fluctuating aspects, but we think that the formal separation into equations (10) and (11) is an aid to clarity.

Let us now apply this approach to the conservative nonlinear growth equation. In order to derive an equation for H, in the frame of reference of the bin, we add the mean velocity to each side of equation (6), and express the current as given by (7) in terms of H using (2). Substituting (7) for $\boldsymbol{j}$ into equation (6), and recalling (3), then leads to

$$\frac{\partial H}{\partial t} = \nu \nabla^2 H + \lambda \nabla \cdot ([H - \overline{H}]\nabla H) + S(\boldsymbol{x}, t). \tag{12}$$

Then following our procedure with the KPZ equation we write the mean and fluctuating equations (analogous to (10) and (11)) as

$$\frac{\partial \overline{H}}{\partial t} = \overline{S} \tag{13}$$

and

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda \boldsymbol{\nabla} \cdot (h \boldsymbol{\nabla} h) + \eta \tag{14}$$

where in both equations we have relied on the spatial homogeneity of the moment $\overline{h\boldsymbol{\nabla} h}$ in order to set its gradient equal to zero. Now we note that the transformation $H \rightarrow H +$ constant is equivalent to $\overline{H} \rightarrow \overline{H} +$ constant, h is unaffected, and equations (12)–(14) are invariant under this transformation.

From equation (13), we see that there is no transient mean velocity when the source term is switched off. This is in contrast to equation (10) for the KPZ model and reflects the conservative nature of (6), in that re-adjustments of the interface height are carried out at constant volume. However, like the KPZ equation as given by (11), both sides of equation (14) average to zero. Similar remarks may apply to other nonlinear growth models for which shift symmetry appears to be absent but, as here, is merely hidden.

Physically, it would be a contradiction in terms to demand shift symmetry directly for h, as this would involve leaving the co-moving frame by means of a symmetry-breaking acceleration. This consideration may be seen as especially relevant to the basic conservative model as given by equation (7).

However, if we replace the KPZ equation by equation (14), then there is a price to pay. The KPZ equation exhibits a form of 'Galilean invariance' whereas (14) does not. In the present context, the significance of Galilean invariance in fluid dynamics is that this symmetry simplifies the application of renormalization group (RG) methods to stirred fluid motion [10]. An analogous simplification has been justified in the case of the KPZ equation, as follows.

Galilean invariance is claimed for the KPZ equation because the change of variable $\boldsymbol{v} = -\boldsymbol{\nabla} h$ transforms it into the Burgers equation, which is known to be Galilean invariant†. However, irrespective of the merits or otherwise of this argument, equation (5) is invariant under an infinitesimal transformation (e.g. see [5])

$$h' = h + \boldsymbol{\epsilon} \cdot \boldsymbol{x} \qquad \boldsymbol{x}' = \boldsymbol{x} - \lambda \boldsymbol{\epsilon} t \qquad \boldsymbol{t} = \boldsymbol{t}' \tag{15}$$

which corresponds to the tilting of the mean interface through a small angle ϵ. This symmetry relies on a cancellation at order $\mathcal{O}(\epsilon)$, and is exactly analogous to Galilean invariance if a term $\mathcal{O}(\epsilon^2)$ is neglected. We shall make two further remarks about this.

First, for the modified form of the KPZ equation given by (11), cancellation of terms at $\mathcal{O}(\epsilon^2)$ makes this an exact symmetry.

Second, it is easily shown that the nonlinear term in equation (14) generates extra terms $\mathcal{O}(\epsilon)$ which break this symmetry. This has consequences for the renormalization group solution of (14), which differs from that of the KPZ equation [9]. Basically, the requirement of Galilean invariance supresses the renormalization of the strength parameter λ in the latter case.

Having said that, we conclude by remarking that perturbative RG in fluid dynamics [10], being restricted to weak coupling, has no application to Navier–Stokes turbulence. Later implementations of RG for turbulence [11, 12], based on iterative conditional averaging, lead to fixed points corresponding to scaling behaviour associated with strong coupling. Such approaches may be relevant to nonlinear growth problems; but ironically their most likely extension would be to the KPZ equation in view of the shared Galilean invariance. An investigation of this point will be the subject of future work.

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† This argument by itself is not totally convincing. Certainly the reverse holds. If an equation for h is Galilean invariant, then an equation for $\boldsymbol{v} = -\nabla h$ is also invariant. However, going from Burgers equation to the KPZ equation involves an integration and so the inference of Galilean invariance does not follow. Note that acceleration is invariant under transformation between inertial frames, but its integral—the velocity—is not.

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